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Comparison of averaging methods for scalar wave propagation in a random elastic layer

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Abstract. The problem of elastic wave propagation in a random layer overlying a homogeneous half-space is treated using (i) an averaging method and (ii) the Born approximation.

The wave is assumed to propagate normally to the layer so that the equations of motion are ordinary differential equations with random coefficients. The two methods are shown to have different ranges of validity; the averaging method applies in the range where stochastic effects are important.

1. Introduction

In Lenoach (1983) we applied the averaging method (Frigerio *et al* 1981) to the problem of surface waves on a random elastic layer. Mainardi *et al* (1980) used the Born approximation in a related one-dimensional model described in § 2. The purpose of this paper is to compare the averaging method, outlined in § 3 and applied to the model in § 4, with the results of the Born approximation (see § 5). The methods are then compared in the region where stochastic effects are known to be important (Papanicolaou 1973, Hersh 1974 and § 6 below) and illustrated by an example in § 7.

2. The model

Following Mainardi *et al* (1980) we consider a plane wave of frequency ω propagating in a layered medium which consists of a homogeneous half-space $z \leqslant 0$ and a heterogenous layer $0 \leqslant z \leqslant H$ (figure 1). The layer density ρ and elastic parameters λ, μ are random functions of z, i.e. they are stochastic processes indexed by z. As in Mainardi *et al* (1980) it is assumed that the displacement components U_j, $j = 1, 2, 3$, are

$$U_j = V_j(z) \exp(-\mathrm{i}\omega t). \tag{2.1}$$

The equations of motion are

$$\begin{aligned} &(\mathrm{d}/\mathrm{d}z)(\mu\, \mathrm{d}V_j/\mathrm{d}z) + \rho\omega^2 V_j = 0 \qquad j = 1, 2 \\ &(\mathrm{d}/\mathrm{d}z)[(\lambda + 2\mu)\, \mathrm{d}V_3/\mathrm{d}z] + \rho\omega^2 V_3 = 0 \end{aligned} \tag{2.2}$$

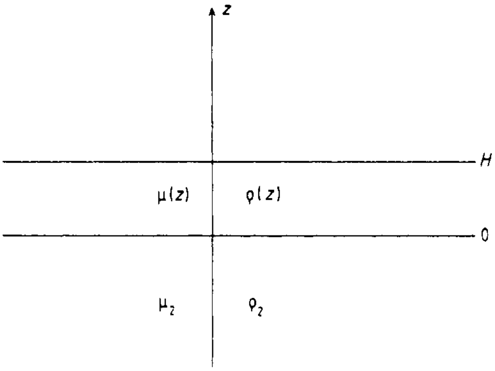

Figure 1. The heterogeneous layer $0 \leqslant z \leqslant H$ and homogeneous half-space $z \leqslant 0$.

with the free-surface boundary condition

$$t(n)_i|_{z=H} = t_{ji}n_j|_{z=H} = 0 \tag{2.3}$$

where $n_j = \delta_{j3}$ is the unit outward normal and $t_{ij} = t_{ji}$ is the Cauchy stress tensor.

Since both the layer and half-space are isotropic, linearly elastic solids we have

$$\begin{aligned} t_{3j} &= \exp(-\mathrm{i}\omega t)\mu(z)\,\mathrm{d}V_j/\mathrm{d}z \qquad j = 1, 2 \\ t_{33} &= \exp(-\mathrm{i}\omega t)(\lambda + 2\mu)(z)\,\mathrm{d}V_3/\mathrm{d}z. \end{aligned} \tag{2.4}$$

Assume

$$\mu(z) = \begin{cases} \mu_2 & z < 0 \\ \mu_0(1 + \varepsilon\mu_1(z)) & 0 \leqslant z \leqslant h \end{cases} \tag{2.5}$$

$$\rho(z) = \begin{cases} \rho_2 & z < 0 \\ \rho_0(1 + \varepsilon\rho_1(z)) & 0 \leqslant z \leqslant H \end{cases} \tag{2.6}$$

where $\mu_2, \rho_2, \mu_0, \rho_0$ are constants and $\rho_1(z), \mu_1(z)$ are stationary, zero-mean, Gaussian stochastic processes with

$$\begin{aligned} &\langle\rho_1(z)\rho_1(z') = R_{\rho\rho}(|z - z'|) \qquad \langle\mu_1(z)\mu_1(z')\rangle = R_{\mu\mu}(|z - z'|) \\ &R_{\mu\rho}(|z - z'| = \langle\mu_1(z)\rho_1(z')\rangle = R_{\rho\mu}. \end{aligned} \tag{2.7}$$

Under these assumptions the equations of motion (2.2) become stochastic differential equations—the dimensionless parameter ε characterises the size of the fluctuations. Note that the correlation functions must satisfy the Schwartz inequality

$$R_{\mu\rho}(z) \leqslant R_{\mu\mu}^{1/2} R_{\rho\rho}^{1/2}. \tag{2.8}$$

All displacement components, because of equation (2.1), may be treated in the same way; for definiteness we focus on V_1. Define $V(z) = V_1(z)$, $t(z) = \mu(z)\,\mathrm{d}V_1/\mathrm{d}z$ and take $V_1(0)$, $t(0)$ to be constants, i.e. assume sure initial data (this assumption is the only difference between our model and the model of Mainardi *et al* (1980). Following Mainardi *et al* we put

$$A = \tfrac{1}{2}V_1(H) \qquad A_a = \tfrac{1}{2}V_a(H) \tag{2.9}$$

where $V_a(z)$ denotes the solution of the homogeneous auxiliary model. The interesting physical quantity is the mean amplification coefficient $\langle A\rangle$ and it must be computed subject to the free surface condition $\langle t\rangle\,(H)=0$.

3. The averaging method

In this section we give a brief description of the averaging method which will be applied to the wave propagation problem. Details may be found in Frigerio *et al* (1981).

The mathematical problem is the following; suppose we are given a differential equation

$$\mathrm{d}f^\lambda(t)/\mathrm{d}t=\lambda A(t)f^\lambda(t) \tag{3.1}$$

on a Banach space B with a projection operator P_0 and intial data $f^\lambda(0)=f_0$ in the subspace $B_0=P_0B$. $\lambda A(t)$ may also be a power series—$\lambda A(t)=\Sigma_{k=1}^{\infty}\lambda^kA_k(t)$. Equation (3.1) has the series solution

$$f^\lambda(t)=\sum_{n=0}^{\infty}\lambda^n\int\ldots\int_{t\geqslant t_1\geqslant\ldots\geqslant t_n\geqslant 0}A(t_1)\ldots A(t_n)\,\mathrm{d}t_n\ldots\mathrm{d}t_1 f_0 \tag{3.2}$$

—the averaging method gives a useful approximation to the projected solution $P_0f^\lambda(t)$ for large values of t as well as a small-time correction term. Assume

$$P_0A(t_1)\ldots A(t_{2m+1})P_0=0 \qquad \forall m=0,1,\ldots \qquad t_1,\ldots,t_{2m+1}\geqslant 0. \tag{3.3}$$

This condition, as we shall see, is satisfied by the model of § 2; to second and fourth order in λ the approximations to $P_0f^\lambda(t)$ are given as

$$\begin{aligned}Y_2^\lambda(t)&=\exp(\lambda^2tG^{(2)})f_0\\ Y_4^\lambda(t)&=(1+\lambda^2M^{(2)}(t))\exp[\lambda^2t(G^{(2)}+\lambda^2G^{(4)})]f_0\end{aligned} \tag{3.4}$$

with estimates, $\mathrm{O}(\lambda^n)$, of the error $\|P_0f^\lambda(t)-Y_n^\lambda(t)\|$ (see Frigerio *et al* 1981). Here $G^{(2)}$, $G^{(4)}$ are time-independent operators on B_0; the operator $M^{(2)}(t)$ gives the short-time correction to asymptotic behaviour.

We are interested in the case when equation (3.1) is a coupled stochastic differential equation; specifically $B_0=IR^2$, $B=L^2\,(\Omega, IR^2, P)$ where (Ω, P) is a probability space. The projection P_0 is then stochastic averaging, i.e. integration over Ω with respect to the measure P.

$$P_0W(t)=\int_\Omega W(t,\omega)P(\mathrm{d}\omega)=\langle W\rangle(t). \tag{3.5}$$

Hereafter, we consider only the second-order approximation $Y_2(t)$; the generator $G^{(2)}$ is

$$G^{(2)}=\lim_{T\to\infty}T^{-1}\int_{t=0}^{T}\left(\int_{S=0}^{t}P_0A_1(t)A_1(s)P_0\,\mathrm{d}s+P_0A_2(t)P_0\right)\mathrm{d}t \tag{3.6}$$

with the error estimate

$$\|P_0f^\lambda(t)-\exp(\lambda^2tG^{(2)})f_0\|\leqslant\lambda^2\beta(\lambda^2t)\sup_{0\leqslant s\leqslant t}\|\exp(\lambda^2G^{(2)}s)f_0\| \tag{3.7}$$

where $\beta(\tau)$ is a positive bounded function of $\tau=\lambda^2t$.

Note that only even powers appear because of equation (3.3); this equation is satisfied by our model because $\mu_1(z)$, $\rho_1(z)$ are zero-mean Gaussian processes so that, for example,

$$\langle\rho_1(z_1)\ldots\rho_1(z_{2n+1})\rangle=0 \qquad \forall n=0,1\ldots \qquad z_1,\ldots,z_{2_n+1}\geqslant 0. \tag{3.8}$$

4. The averaging method applied to the model

Put $p(z)=\binom{V(z)}{t(z)}$ so that equations (2.2), (2.4) give

$$\mathrm{d}p/\mathrm{d}z=[L_0+\varepsilon L_1(z)+\varepsilon^2L_2(z)]p(z) \tag{4.1}$$

where

$$L_0=\begin{pmatrix}0 & 1\\ -k_0^2 & 0\end{pmatrix} \qquad L_1=-\begin{pmatrix}0 & \mu_1(z)\\ k_0^2\rho_1(z) & 0\end{pmatrix}$$
$$L_2=\begin{pmatrix}0 & \mu_1^2(z)\\ 0 & 0\end{pmatrix} \qquad k_0^2=\omega^2\rho_0/\mu_0=\omega^2/\beta^2. \tag{4.2}$$

Define

$$U_z=\exp(L_0z)=\begin{pmatrix}\cos k_0z & (1/k_0)\sin k_0z\\ -k_0\sin k_0z & \cos k_0z\end{pmatrix} \tag{4.3}$$

and $q(z)=U_{-z}(z)$ so that

$$\mathrm{d}q/\mathrm{d}z=[\epsilon A_1(z)+\epsilon^2A_2(z)]q(z)$$
$$q(0)=p(0) \qquad A_i(z)=U_{-z}L_i(z)U_z \qquad i=1,2. \tag{4.4}$$

Equation (4.4) is a differential equation of the form (3.1) with z as the 'time' variable; hence the results of § 3 may be applied directly. We have

$$\langle p(z)\rangle=U_z\exp(\varepsilon^2zG^{(2)})p(0) \tag{4.5}$$

with $G^{(2)}$ given by equation (3.6). Let $S_{\mu\mu}(\beta)=\int_0^\infty\exp(\mathrm{i}\beta r)R_{\mu\mu}(r)\,\mathrm{d}r$ and define $S_{\mu\rho}$, $S_{\rho\rho}$ similarly. Then

$$\langle V_1(z)\rangle=\exp(\gamma z)[\cos\alpha z\; V_1(0)+k_0\mu_0\sin\alpha z\; t(0)] \tag{4.6}$$

$$\langle t(z)\rangle=\exp(\gamma z)\,[\cos\alpha z\; t(0)-k_0\mu_0V_1(0)\sin\alpha z] \tag{4.7}$$

where

$$\gamma(\omega)=-(\varepsilon^2k_0^2/H)\{[S_{\mu\mu}(0)-\mathrm{Re}\,S_{\mu\mu}(2k_0)]$$
$$+[S_{\rho\rho}(0)-\mathrm{Re}\,S_{\rho\rho}(2k_0)]-2[S_{\mu\rho}(0)+\mathrm{Re}\,S_{\mu\rho}(2k_0)]\} \tag{4.8}$$

$$\alpha(\omega)=k_0+\tfrac{1}{2}\varepsilon^2k_0\{R_{\mu\mu}(0)-\tfrac{1}{2}k_0[\mathrm{Im}\,S_{\mu\mu}(2k_0)+\mathrm{Im}\,S_{\rho\rho}(2k_0)+2\,\mathrm{Im}\,S_{\mu\rho}(2k_0)]\}. \tag{4.9}$$

The boundary condition $\langle t\rangle(H)=0\Rightarrow$

$$\langle V_1(z)\rangle=V_1(0)\exp(\gamma z)\cos\alpha(z-H)/\cos\alpha H$$
$$\langle t(z)\rangle=-k_0\mu_0V_1(0)\sin\alpha(z-H)/\cos\alpha H. \tag{4.10}$$

Since the solution of the homogeneous auxiliary model is $V_a(z) = (V_1(0)/\cos k_0H)\cos k_0(z-H)$, the final result is

$$\langle A\rangle = (A_a \cos k_0H/\cos \alpha H)\exp(\gamma H). \tag{4.11}$$

$\alpha(\omega)$, $\gamma(\omega)$ may be though of as the phase and amplitude effects, respectively, of the fluctuations introduced via equations (2.5) and (2.6).

5. The Born approximation

In Mainardi *et al* (1980) the method used to find a solution correct to $O(\epsilon^2)$ is the Born approximation. The differential equation

$$(1\mathrm{d}/\mathrm{d}z - L_0)p(z) = (\epsilon L_1(z) + \epsilon^2 L_2(z))p(z) \tag{5.1}$$

is converted into an integral equation

$$p(z) = p_a(z) + \epsilon \int_0^H G(z, z')W(z')p(z')\,\mathrm{d}z' \tag{5.2}$$

where $G(z, z')$ is the Green's matrix of equation (5.1), $p_a(z) = U_z p(0)$ is the homogeneous solution and $W = (L_1 + \varepsilon L_2)$. Iterating this integral equation twice and averaging the result yields the second Born approximation

$$\langle p(z)\rangle = p_a(z) + \epsilon^2 \int_0^H \mathrm{d}z'\, G(z, z')\Big(\langle L_2(z')\rangle p_a(z') + \int_0^H \mathrm{d}z''\, M(z', z'')p_a(z'')\Big) \tag{5.3}$$

where

$$M(z, z') = \langle L_1(z)G(z, z')L_1(z')\rangle \tag{5.4}$$

and we have used $\langle L_1(z)\rangle = 0$.

The Green's matrix $G(z, z')$ is defined as the solution of

$$(1\mathrm{d}/\mathrm{d}z - L_0)G(z, z') = \delta(z - z')1 \qquad G(0, z') = 0 \tag{5.5}$$

where the initial condition is chosen to give sure initial data, $\langle p(0)\rangle = p(0)$. Therefore

$$G(z, z') = \begin{cases} 0 & 0 \leqslant z \leqslant z' \\ U_{z-z'} & z' \leqslant z \leqslant H \end{cases} \tag{5.6}$$

giving

$$\langle p\rangle(z) = p_a(z) + \epsilon^2\Big(\int_0^z \mathrm{d}z'\, U_{z-z'}\langle L_2(z')\rangle U_{z'} + \int_0^z \mathrm{d}z' \int_0^{z'} \mathrm{d}z''\, U_{z-z'}\langle L_1(z')U_{z'-z''}L_1(z'')\rangle U_{z''}\Big)p(0) \tag{5.7}$$

or, in the notation of § 3, § 4,

$$\langle p\rangle(z) = U_z\Big[1 + \epsilon^2\Big(\int_0^z \mathrm{d}z'\, P_0A_2(z')P_0 + \iint_{z\geqslant z'\geqslant z''\geqslant 0} P_0A_1(z')A_1(z'')P_0\,\mathrm{d}z'\,\mathrm{d}z''\Big)\Big]p(0). \tag{5.8}$$

Comparison with equation (3.2) shows that equation (5.8) is simply the result of truncating the series solution (3.2), modified to include the $A_2(t)$ term, at $O(\varepsilon^2)$.

6. The second-order limit

We are concerned with problems involving differential equations in which the random terms are multiplied by small coefficients. In such circumstances random terms are important only if they lead to cumulative effects over long times. Mathematically this is expressed by a balance between two limits: small fluctuations and large times. For a stochastic equation such as (3.1) the second-order limit is $\lambda \to 0$, $t \to \infty$ with $\lambda^2 t = \tau$ fixed; so that t is of order $1/\lambda^2$. Papanicolaou (1973) treats this point exhaustively in his review which emphasises applications of the theory: a review from the standpoint of probability theory is given by Hersh (1974).

Papanicolaou and Keller (1971) used a formal 'two-time' method to study the random harmonic oscillator: in Papanicolaou *et al* (1973) Has'minskii's limit theorem is applied to the problem of a beam in a strongly focusing random medium. The usefulness of limit theorems is restricted in practice because they require the explicit solution of a diffusion equation and this is rarely possible: it is preferable to approach physical problems from the standpoint of differential equations (Frigerio *et al* 1981)

Returning to the Born approximation equation (5.8) gives

$$\langle p\rangle(H) = U_H[1+\varepsilon^2 B]p(0) \tag{6.1}$$

where

$$B = \int_0^H \mathrm{d}z\, U_{-z}\Big(\langle L_2(z)\rangle U_z + \int_0^z \mathrm{d}z'\, \langle L_1(z)U_{z-z'}L_1(z')\rangle U_{z'}\Big). \tag{6.2}$$

The second-order limit in this problem is $\varepsilon \to 0$, $H \to \infty$ with $\varepsilon^2 H$ fixed—in this limit the Born approximation gives

$$\lim_{\substack{\varepsilon\to 0\\ H\to\infty}} \varepsilon^2 B = \varepsilon^2 H \lim_{H\to\infty} H^{-1} \int_0^H \mathrm{d}z\, U_{-z}\Big(\langle L_2(z)\rangle U_z + \int_0^z \mathrm{d}z'\, \langle L_1(z)U_{z-z'}L_1(z')\rangle U_{z'}\Big)$$

$$= \varepsilon^2 H G^{(2)}. \tag{6.3}$$

The mean amplification coefficient $\langle A\rangle$ is then

$$\langle A\rangle = A_a(1+\gamma H + \bar{\alpha} H \tan k_0 H) \tag{6.4}$$

where $\bar{\alpha} = \alpha - k_0$. This approximation may also be obtained by expanding the result of equation (4.11) and neglecting terms of order τ^2, $\tau = \varepsilon^2 H$.

7. Discussion of the results

Let ν be a typical correlation length of the layer, for example $\nu = \max(\nu_{\mu\mu}, \nu_{\mu\rho}, \nu_{\rho\rho})$ where $\nu_{\rho\rho} = \int_0^\infty R_{\rho\rho}(r)\,\mathrm{d}r$ etc, and H' be the dimensionless length $H' = H/\nu$. To illustrate equations (4.11), (6.4) we examine the dimensionless ratio $\langle |A|\rangle/|A_a|$ in the case of stochastic fluctuations in μ only with Gaussian correlation function

$$R_{\mu\mu} = \exp\{-[(z-z')/\nu]^2\} \tag{7.1}$$

which gives

$$\gamma H = -\tfrac{1}{8}\varepsilon^2 H^1\sqrt{\pi}(k_0\nu)^2[1-\exp(-k_0^2\nu^2)] \tag{7.2}$$

$$\tilde{\alpha} H = \varepsilon^2 H^1(k_0\nu/2)\left(1-(k_0\nu/2)\int_0^{k_0\nu}\exp(r^2-k_0^2\nu^2)\,\mathrm{d}r\right). \tag{7.3}$$

$\langle|A|\rangle/|A_a|$ is, thus, a function of the dimensionless variable $k_0\nu$ with ε, H' as dimensionless parameters restricted by the conditions $\varepsilon^2 \ll 1$, $H' \gg 1$ with $\varepsilon^2 H'$ of order one. Figure 2 shows that the two methods give significantly different behaviour and that the Born approximation is unsatisfactory for intermediate values of $k_0\nu$.

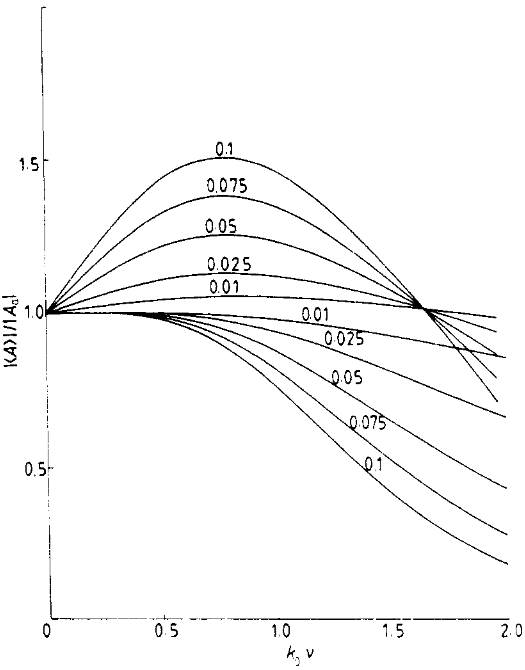

Figure 2. The amplification ratio $\langle|A|\rangle/|A_a|$ for the correlation functions (7.1): the upper and lower curves are given by the Born approximation and the averaging method respectively. Numerical values refer to ε^2: $H' = 20$,

To summarise, we have treated a slightly modified version of the problem discussed in Mainardi *et al* (1980) focusing on the mean amplification coefficient in the region where stochastic effects are appreciable (expressed mathematically by the second-order limit). Indeed, the approximation $Y_2^\lambda(t) = \exp[\lambda^2 tG^{(2)}]f_0$ becomes exact in this limit since if $\lambda^2\sigma = s$, $\lambda^2 t = \tau$ equation (3.7) is

$$\|P_0 f^\lambda(\tau/\lambda^2) - \exp[\tau G^{(2)}]f_0\| \leq \lambda^2\beta(\tau)\sup_{0\leq\sigma\leq\tau}\|\exp[\sigma G^{(2)}]f_0\| \tag{7.4}$$

so that

$$\lim_{\lambda\to 0}\|P_0 f^\lambda(\tau/\lambda^2) - \exp[\tau G^{(2)}]f_0\| = 0. \tag{7.5}$$

There is no comparable estimate of the error in terms of λ and t for the Born approximation—the results of this paper suggest that in the region where stochastic effects produce interesting results it is an unreliable technique.

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